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Five-Dimensional Axisymmetric Spacetime Solutions

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We give a method by which a class of 5D potential solutions and a class of 5D spacetime solutions with magnetic field can be generalized, and show some examples.

1. INTRODUCTION

Five-dimensional (5D) gravitational theory was proposed by Kaluza and Klein in the 1920s. For the 5D axially symmetric case, some stationary solutions with asymptotically flat behavior have been obtained (Belinskey and Ruffini, 1980; Clement, 1986). By considering the Bonnor solution to the 4D Maxwell–Einstein equations a 5D solution with magnetic and electric fields was given by Becerril and Matos (1990). By using the potential formalism Becerril and Matos (1992a,b; Matos, 1994a,b) found some axisymmetric solutions with magnetic field, one of which has Schwarzschild-like behavior. Matos (1994a,b) discussed 5D stationary axisymmetric solutions as harmonic maps. For 4D Maxwell–Einstein equations Manko and Sibgatullin (1992) and Chamorro *et al.* (1991) obtained solutions with electric or magnetic field, and recently Ruiz *et al.* (1995) found *N*-soliton solutions.

In this paper, we give a method by which a class of 5D potential solutions corresponding to the axisymmetric spacetime solutions with electric field and a class of the axisymmetric solutions with magnetic field can be generalized, we give some explicit examples. The first is a 5D potential solution with ST-like property; the second is a static 5D spacetime solution; and the third is obtained from the new solution given by Matos.

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Consider the following Lagrangian:

$$\mathcal{L} = (\alpha/2f^2)[f_{,i}f^{,i} + (\epsilon_{,i} + \psi\chi_{,i})(\epsilon^{,i} + \psi\chi^{,i})] + (\alpha/2f)[\kappa^2\psi_{,i}\psi^{,i} + \kappa^{-2}\chi_{,i}\chi^{,i}] + \tfrac{2}{3}\alpha\kappa^{-2}\kappa_{,i}\kappa^{,i} \qquad (1.1)$$

Then a 5D potential space metric is defined as

$$ds^2 = (1/2f^2)[df^2 + (d\epsilon + \psi\, d\chi)^2] + (1/2f)[\kappa^2\, d\psi^2 + \kappa^{-2}\, d\chi^2] + \tfrac{2}{3}\kappa^{-2}\, d\kappa^2 \qquad (1.2)$$

where $(f, \epsilon, \kappa, \psi, \chi)$ are gravitational, rotational, scalar, electric, and magnetic potentials. We define a 3×3 matrix $M = (M_{ij})$ with unit determinant

$$\begin{aligned}
M_{11} &= -2f^{-1}\kappa^{-2/3}(f^2 + \epsilon^2 - f\kappa^2\psi^2)\\
M_{22} &= -2f^{-1}\kappa^{-2/3}\\
M_{33} &= (-1/4)f^{-1}\kappa^{-2/3}(\chi^2 - \kappa^2 f)\\
M_{12} &= M_{21} = 2f^{-1}\kappa^{-2/3}\epsilon\\
M_{13} &= M_{31} = (1/\sqrt{2})f^{-1}\kappa^{-2/3}(\epsilon\chi + f\kappa^2\psi)\\
M_{23} &= M_{32} = (-1/\sqrt{2})f^{-1}\kappa^{-2/3}\chi
\end{aligned} \qquad (1.3)$$

For the axisymmetric case, the 3×3 matrix M satisfies the following equation:

$$(\rho M_{,\rho}M^{-1})_{,\rho} + (\rho M_{,z}M^{-1})_{,z} = 0 \qquad (1.4)$$

For 5D axisymmetric spacetime, the metric reads

$$ds^2 = F(d\rho^2 + dz^2) + G_{ij}\, dx_i\, dx_j, \qquad i, j = 3, 4, 5 \qquad (1.5)$$

Define a 3×3 matrix $G = (G_{ij})$ with the determinant $\det G = -\rho^2$. Then the vacuum field equation reduces to

$$(\rho G_{,\rho}G^{-1})_{,\rho} + (\rho G_{,z}G^{-1})_{,z} = 0 \qquad (1.6)$$

The metric coefficient F is determined by

$$(\ln F)_{,\rho} = -\rho^{-1} + (4\rho)^{-1}\,\mathrm{tr}(U^2 - V^2) \qquad (1.7a)$$

$$(\ln F)_{,z} = (2\rho)^{-1}\,\mathrm{tr}\, UV \qquad (1.7b)$$

$$U = \rho G_{,\rho}G^{-1}, \qquad V = \rho G_{,z}G^{-1} \qquad (1.7c)$$

2. GENERATING 5D AXISYMMETRIC SOLUTIONS

For equations (1.4) and (1.6), we have the following theorem.

Consider a 3×3 matrix $R = (R_{ij})$

$$R = \begin{vmatrix} R_{11} & 0 & R_{13} \\ 0 & R_{22} & 0 \\ R_{31} & 0 & R_{33} \end{vmatrix} \tag{2.1}$$

We construct a new 3×3 matrix $\overline{R} = (\overline{R}_{ij})$ as follows:

$$\overline{R} = \begin{vmatrix} \lambda R_{11} & 0 & \lambda R_{13} \\ 0 & \lambda^{-2} R_{22} & 0 \\ \lambda R_{31} & 0 & \lambda R_{33} \end{vmatrix} \tag{2.2}$$

with the function $\lambda = e^{\Psi}$, where Ψ satisfies the Laplace equation

$$\Psi_{,\rho\rho} + \rho^{-1}\Psi_{,\rho} + \Psi_{,zz} = 0 \tag{2.3}$$

If R is a solution to equation (1.4) or (1.6), then the matrix $\overline{R}$ is a solution to equation (1.4) or (1.6).

One can directly prove this theorem by equation (1.4) or (1.6). The theorem can be applied to the following two cases.

(a) Rotational potential $\epsilon = 0$, magnetic potential $\chi = 0$. In this case a potential solution corresponds to a 5D spacetime solution with electric field.

(b) Spacetime metric coefficients $G_{34} = G_{43} = 0$, $G_{45} = G_{54} = 0$. In this case the metric (1.5) describes a 5D spacetime with magnetic field.

3. EXAMPLE AND APPLICATION

(a) Define the 3×3 matrix

$$M = (1/H_J)\begin{vmatrix} K_J^2 + (-1)^J H_J^2 & 0 & K_J \\ 0 & (-1)^J H_J & 0 \\ K_J & 0 & 1 \end{vmatrix} \tag{3.1}$$

where

$$H_J = [p_J^2 x^2 + (-1)^J q_J^2 y^2 - 1]/[(p_J x + 1)^2 + (-1)^J q_J^2 y^2] \tag{3.2a}$$

$$K_J = 2q_J y/[(p_J x + 1)^2 + (-1)^J q_J^2 y^2] \tag{3.2b}$$

$$x = (1/2k)\{[\rho^2 + (z + k)^2]^{1/2} + [\rho^2 + (z - k)^2]^{1/2}\} \tag{3.2c}$$

$$y = (1/2k)\{[\rho^2 + (z + k)^2]^{1/2} - [\rho^2 + (z - k)^2]^{1/2}\} \tag{3.2d}$$

$$p_J^2 + (-1)^J q_J^2 = 1 \tag{3.2e}$$

and where $J = 1, 2$ and k, p, and q are constants. We call (3.1) the double 5D TS-like potential solution to equation (1.4) (Tomimatsu and Sato, 1972;

Zhong, 1985). According to the above theorem, the TS-like solution (3.1) can be generalized as

$$M = (1/H_J)e^{\Psi}\begin{vmatrix} K_J^2 + (-1)^J H_J^2 & 0 & K_J \\ 0 & (-1)^J e^{-3\Psi} H_J & 0 \\ K_J & 0 & 1 \end{vmatrix} \tag{3.3}$$

(b) From the 5D spacetime solution to (1.6)

$$G = \mathrm{diag}(\rho^2 e^{\Phi}, -1, e^{-\Phi}) \tag{3.4}$$

where the function Φ satisfies the Laplace equation

$$\Phi_{,\rho\rho} + \rho^{-1}\Phi_{,\rho} + \Phi_{,zz} = 0$$

according to the theorem we obtain

$$G = \mathrm{diag}(\rho^2 e^{\Psi+\Phi}, -e^{-2\Psi}, e^{\Psi-\Phi}) \tag{3.5}$$

(c) We write the 5D spacetime metric with magnetic field

$$ds^2 = F^{(5)}(d\rho^2 + dz^2) + I^{-2}\rho^2\, d\varphi^2 - dt^2 + I^2(A_3\, d\varphi + dx^5)^2 \tag{3.6}$$

where the metric coefficients read

$$\begin{aligned} G_{33} &= I^{-2}\rho^2 + I^2 A_3^2 \\ G_{44} &= -1, \qquad G_{55} = I^2 \\ G_{35} &= G_{53} = I^2 A_3 \end{aligned} \tag{3.7}$$

According to the above theorem, the metric (3.6) is

$$ds^2 = \overline{F}^{(5)}(d\rho^2 + dz^2) + e^{\Psi}[I^{-2}\rho^2\, d\varphi^2 - e^{-3\Psi}\, dt^2 + I^2(A_3\, d\varphi + dx^5)^2] \tag{3.8}$$

where the metric coefficient $\overline{F}^{(5)}$ is determined by $F^{(5)}$ and Ψ,

$$\ln(\overline{F}^{(5)}/F^{(5)})_{,\rho} = (3\rho/2)[(\Psi_{,\rho})^2 - (\Psi_{,z})^2] + \Psi_{,\rho} \tag{3.9a}$$

$$\ln(\overline{F}^{(5)}/F^{(5)})_{,z} = 3\rho(\Psi_{,\rho})(\Psi_{,z}) + \Psi_{,z} \tag{3.9b}$$

Consider the asymptotically flat spacetime metric (Matos, 1992a) with metric coefficients given as follows:

$$\begin{aligned} G_{55} &= I^2 = 2\{2 - bm^2(\cos\theta)/[(r-m)^2 - m^2\cos^2\theta]\}^{-1} \\ G_{35} &= G_{53} = I^2 A_3 = I^2(\sqrt{2}m^2/4)(r-m)(\sin^2\theta)/[(r-m)^2 - m^2\cos^2\theta] \\ G_{33} &= I^{-2}\rho^2 + I^2 A_3^2 \\ G_{44} &= -1 \\ F^{(5)} &= I^{-2} \end{aligned} \tag{3.10}$$

where $b = \pm(1/\sqrt{2})$ and

$$\rho = (r^2 - 2mr)^{1/2} \sin \theta, \qquad z = (r - m) \cos \theta \tag{3.11}$$

Furthermore, choose

$$\Psi = \Phi = c(\rho^2 + z^2)^{-1/2} \tag{3.12}$$

where c is a constant. Then a new asymptotically flat spacetime is explicitly exhibited:

$$\begin{aligned}
\overline{G}_{55} &= I^2 e^{\Phi} = 2\{2 - bm^2(\cos\theta)/[(r - m)^2 - m^2\cos^2\theta]\}^{-1} e^{\Phi} \\
\overline{G}_{35} &= \overline{G}_{53} = I^2 A_3 e^{\Phi} = I^2 e^{\Phi}(\sqrt{2}m^2/4)(r - m)(\sin^2\theta)/[(r - m)^2 \\
&\quad - m^2 \cos^2\theta] \qquad (3.13) \\
\overline{G}_{33} &= (I^{-2}\rho^2 + I^2 A_3^2) e^{\Phi} \\
\overline{G}_{44} &= -e^{-2\Phi} \\
\ln(\overline{F}^{(5)} I^2) &= (-3/4)c^{-2}\rho^2\Phi^4 + \Phi
\end{aligned}$$

Similarly, all other solutions given in Matos (1992a,b) can be generalized.

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